

VERTEX CORRECTIONS IN GAUGE THEORIES FOR TWO-DIMENSIONAL CONDENSED MATTER SYSTEMS

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We calculate the self-energy of two-dimensional fermions that are coupled to transverse gauge fields, taking two-loop corrections into account. Given a bare gauge field propagator that diverges for small momentum transfers q as $1/q^\eta$, $1 < \eta \leq 2$, the fermionic self-energy without vertex corrections vanishes for small frequencies ω as $\Sigma(\omega) \propto \omega^\gamma$ with $\gamma = \frac{2}{1+\eta} < 1$. We show that inclusion of the leading radiative correction to the fermion - gauge field vertex leads to $\Sigma(\omega) \propto \omega^\gamma [1 - a_\eta \ln(\omega_0/\omega)]$, where a_η is a positive numerical constant and ω_0 is some finite energy scale. The negative logarithmic correction is consistent with the scenario that higher order vertex corrections push the exponent γ to larger values.

1. Introduction

The problem of two-dimensional fermions that are coupled to a transverse gauge field has recently received a lot of attention, because it arises as effective low-energy theory in two different physical contexts. On the one hand, the infrared physics of the two-dimensional $t - J$ -model has been argued to be correctly described by fermions and bosons that are coupled to an Abelian gauge field^{1,2,3}. The other example are half-filled quantum Hall systems⁴, where a fictitious Chern-Simons gauge field can be used to attach two quanta of a magnetic flux to the physical electron, thus forming a new (presumably stable) quasi-particle, the so-called composite fermion⁵. Within mean field theory, where fluctuations of the gauge field are ignored, the fermions are assumed to form a conventional Fermi liquid. Many authors have considered the stability of the Fermi liquid with respect to fluctuations of the gauge field. When one calculates the fermionic self-energy $\Sigma(\omega)$ to first order in the dynamically screened propagator of the gauge field, one finds for small frequencies^{4,6,7,8}

$$\Sigma(\omega) \propto \omega^\gamma , \quad (1)$$

with $\gamma < 1$. This implies a vanishing quasi-particle residue $Z = \lim_{\omega \rightarrow 0} [1 - \partial \Sigma(\omega) / \partial \omega]^{-1}$, so that the system cannot be a Fermi liquid. Because first order perturbation theory qualitatively changes the analytic structure of the non-interacting

Green's function and there is no obvious small parameter in the problem, it is necessary to address the effect of higher orders in perturbation theory.

Clearly, in order to understand the physics of gauge fields in these strongly correlated systems, conventional perturbative many-body theory is not sufficient. At least one should sum properly chosen infinite sub-classes of Feynman diagrams. Over the past few years a variety of methods have been proposed to resum the most divergent terms to all orders in perturbation theory^{7,8,9,10,11,12,13,14,15,16,17}. However, so far a general agreement has not been achieved. In particular, in Ref.¹⁷ it was found by means of a non-perturbative functional integral approach^{18,19} that higher orders completely change the scenario suggested by a one-loop calculation: the non-analyticities suggested by lowest order perturbation theory were found to be partially removed, so that the spectral function exhibits a well-defined quasi-particle peak. This implies that there must exist some higher order terms in the perturbative expansion of the self-energy that are more singular than the one-loop result. In this work we shall explicitly calculate the two-loop correction to the self-energy due to the leading radiative correction to the fermion - gauge field vertex, and show that it contains an additional factor proportional to $\ln(\omega_0/\omega)$ as compared with the one-loop result (ω_0 is some finite frequency, see Eq.(72) below). For reasons that will be explained in detail in Sec.3, this logarithmic correction has been missed in a previous calculation by Altshuler, Ioffe and Millis⁸. The existence of a logarithmically divergent correction implies that the true infrared behavior of the fermionic self-energy cannot be obtained from a one-loop calculation. In fact, the prefactor of the logarithm turns out to be negative (relative to the one-loop self-energy, see Eq.(72)), so that it is consistent with the scenario that the summation of vertex corrections to infinite orders leads to an *increase* of the exponent γ in Eq.(1), and possibly restores Fermi liquid behavior¹⁷ (which requires $\gamma \geq 1$).

2. The self-energy without vertex corrections

Let us begin with a careful discussion of the one-loop fermionic self-energy correction Σ_1 due to fluctuations of the gauge field. The relevant Feynman diagram is shown in Fig.1. Although this diagram has been calculated previously by many authors^{4,6,7,8,14} let us calculate it once more, using a particular coordinate system where all wave-vectors are measured relative to a fixed point on the Fermi surface. The insights gained from this calculation will be useful for the more difficult two-loop calculation. Using the Matsubara formalism, Fig.1 represents the following expression for the fermionic self-energy,

$$\Sigma_1^\alpha(\tilde{k}) = -\frac{1}{\beta V} \sum_{\tilde{q}} h_{\tilde{q}}^{\text{RPA},\alpha} G_0^\alpha(\tilde{k} + \tilde{q}), \quad (2)$$

where V is the volume of the system, β is the inverse temperature, and we have defined collective labels $\tilde{k} = [\mathbf{k}, i\tilde{\omega}_n]$, $\tilde{q} = [\mathbf{q}, i\omega_n]$, where $\tilde{\omega}_n = 2\pi(n + \frac{1}{2})/\beta$ are fermionic Matsubara frequencies, and $\omega_n = 2\pi n/\beta$ are bosonic ones. The superscript α indicates that all wave-vectors are measured with respect to a point \mathbf{k}^α on

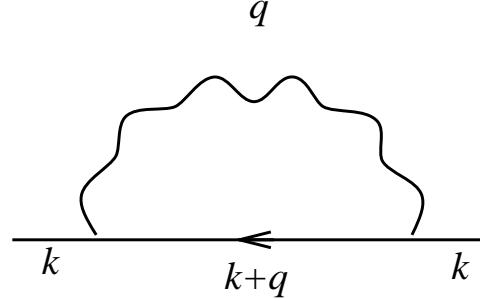


Fig. 1. Leading fluctuation correction to the fermionic self-energy. The solid arrow is the mean-field Green's function, and the wavy-line is the RPA screened propagator of the gauge field, see Eqs.(3) and (4).

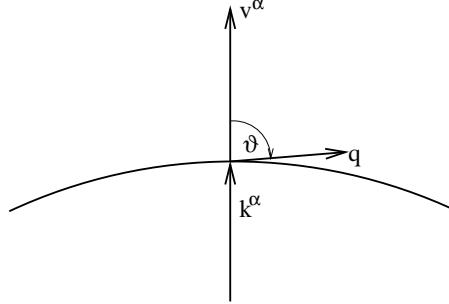


Fig. 2. Definition of circular coordinates centered at point \mathbf{k}^α on the Fermi surface.

the Fermi surface, as shown in Fig.2. The free Green's function is

$$G_0^\alpha(\tilde{\mathbf{k}}) = \frac{1}{i\tilde{\omega}_n - \xi_{\mathbf{k}^\alpha + \mathbf{k}}} , \quad (3)$$

where $\xi_{\mathbf{k}^\alpha + \mathbf{k}} = \mathbf{v}^\alpha \cdot \mathbf{k} + \mathbf{k}^2/(2m^\alpha)$ is the mean field energy dispersion (measured relative to the chemical potential) for wave-vectors \mathbf{k} close to \mathbf{k}^α . Here \mathbf{v}^α is the local Fermi velocity with magnitude v_F , and m^α is the local effective mass close to \mathbf{k}^α . Note that we are not assuming that the Fermi surface is spherically symmetric; in particular, for $m^\alpha \rightarrow \infty$ the Fermi surface becomes locally flat. The dynamically screened gauge field propagator is within the random phase approximation (RPA) and for frequencies in the regime $|\omega_n| \lesssim v_F q$ given by

$$h_{\tilde{q}}^{\text{RPA},\alpha} = -\frac{2\pi}{m^*} [1 - (\hat{\mathbf{v}}^\alpha \cdot \hat{\mathbf{q}})^2] \frac{v_F q}{\Gamma_q + |\omega_n|} , \quad (4)$$

where we have used the Coulomb gauge, $\hat{\mathbf{v}}^\alpha$ is a unit vector parallel to \mathbf{v}^α , and the energy scale Γ_q is

$$\Gamma_q = v_F q (q/q_c)^\eta = v_F q_c (q/q_c)^{1+\eta} . \quad (5)$$

The mass m^* is some effective mass such that $m^* v_F \equiv k_F$ is a measure for the average curvature of the Fermi surface. In contrast, $m^\alpha v_F \equiv k_c$ measures the *local* curvature of the Fermi surface. Throughout this work we shall assume that the

momentum scale q_c in Eq.(5) is small compared with k_F and k_c , and that the exponent η is in the interval $1 < \eta \leq 2$. The case $\eta = 2$ is relevant to gauge theories of high-temperature superconductors, as well as half-filled quantum Hall systems with short-range density-density interactions.

To perform the integration over the momentum-transfer \mathbf{q} in Eq.(2), we choose the circular coordinates shown in Fig.2. For simplicity, we shall restrict ourselves to external wave-vectors of the form $\mathbf{k} = k_{\parallel} \hat{\mathbf{v}}^{\alpha}$, so that

$$\xi_{\mathbf{k}^{\alpha} + \mathbf{k} + \mathbf{q}} = \xi_{k_{\parallel}} + (1 + \frac{k_{\parallel}}{k_c}) v_F q \cos \vartheta + \frac{q^2}{2m^{\alpha}}, \quad (6)$$

where $\xi_{k_{\parallel}} = v_F k_{\parallel} + k_{\parallel}^2 / (2m^{\alpha})$. Keeping in mind that the form (4) for the gauge field propagator is valid for $|\omega_n| \lesssim v_F q$, and imposing an ultraviolet cutoff κ on the q -integration in Eq.(2) (anticipating that the leading behavior for small frequencies is dominated by the infrared singularities of the integrand, we may choose $q_c \ll \kappa \ll k_F$) we obtain from Eq.(2) for $V \rightarrow \infty$ and $\beta \rightarrow \infty$,

$$\begin{aligned} \Sigma_1^{\alpha}(k_{\parallel}, i\tilde{\omega}_n) &= \frac{1}{(2\pi)^2 m^*} \int_0^{\kappa} dq q \int_{-v_F q}^{v_F q} d\omega \frac{v_F q}{|\omega| + \Gamma_q} \\ &\times \int_{-\pi}^{\pi} d\vartheta \frac{\sin^2 \vartheta}{G_0^{-1} + i\omega - (1 + \frac{k_{\parallel}}{k_c}) v_F q \cos \vartheta - \frac{q^2}{2m^{\alpha}}}, \end{aligned} \quad (7)$$

with $G_0^{-1} = i\tilde{\omega}_n - \xi_{k_{\parallel}}$. Scaling out a factor of $v_F q (1 + \frac{k_{\parallel}}{k_c})$ and defining the complex variable

$$W(q, \omega) = \frac{G_0^{-1} + i\omega - \frac{q^2}{2m^{\alpha}}}{v_F q (1 + \frac{k_{\parallel}}{k_c})}, \quad (8)$$

Eq.(7) can also be written as

$$\Sigma_1^{\alpha}(k_{\parallel}, i\tilde{\omega}_n) = \frac{1}{2\pi m^*(1 + \frac{k_{\parallel}}{k_c})} \int_0^{\kappa} dq q \int_{-v_F q}^{v_F q} d\omega \frac{Z(W(q, \omega))}{|\omega| + \Gamma_q}. \quad (9)$$

Here the complex function $Z(W)$ is defined by²⁰

$$Z(W) = \frac{1}{\pi} \int_0^{\pi} d\vartheta \frac{\sin^2 \vartheta}{W - \cos \vartheta} = W - \sqrt{W^2 - 1}, \quad (10)$$

where the root has to be taken such that $|Z| < 1$. It turns out that the leading infrared behavior of Eq.(9) is due to the regime where $|W(q, \omega)| \ll 1$. To see this, note that

$$Z(W) = \begin{cases} -i\text{sgn}(\text{Im}W) + W + O(W^2) & \text{if } |W| \ll 1 \\ (2W)^{-1} + O(W^{-2}) & \text{if } |W| \gg 1 \end{cases}. \quad (11)$$

Because $|\omega|/(v_F q) \lesssim 1$ and $\frac{q^2}{2m^{\alpha}} \lesssim v_F q$ in the domain of integration in Eq.(9), the condition $|W| \ll 1$ is equivalent with

$$q \gtrsim k_0 \equiv \max \{ |k_{\parallel}|, |\tilde{\omega}_n|/v_F \}. \quad (12)$$

To evaluate the integral in Eq.(9), we subdivide the q -integration into the regimes $0 < q < k_0$ and $k_0 < q < \kappa$, and use the corresponding asymptotic forms of $Z(W)$ given in Eq.(11). Using the fact that $|Z(W)| \leq 1$, it is not difficult to show that the contribution from the regime $0 < q < k_0$ is

$$\Sigma_1^\alpha(k_\parallel, i\tilde{\omega}_n)_{(q < k_0)} \propto \frac{k_0^2}{m^*} \ln(q_c/k_0) . \quad (13)$$

Obviously, for small $\tilde{\omega}_n$ and k_\parallel this correction is negligible. On the other hand, in the regime $k_0 < q < \kappa$ we may approximate (see Eqs.(8) and (11))

$$Z(W(q, \omega)) \approx -i\text{sgn}(\tilde{\omega}_n + \omega) . \quad (14)$$

The corresponding contribution to the self-energy is for $|k_\parallel| \ll k_c$

$$\Sigma_1^\alpha(k_\parallel, i\tilde{\omega}_n)_{(q > k_0)} = \frac{-i}{2\pi m^*} \int_{k_0}^\kappa dq q \int_{-v_F q}^{v_F q} d\omega \frac{\text{sgn}(\tilde{\omega}_n + \omega)}{|\omega| + \Gamma_q} . \quad (15)$$

Using the fact that the rest of the integrand is an even function of ω , we may replace under the integral sign

$$\text{sgn}(\tilde{\omega}_n + \omega) \rightarrow \text{sgn}(\tilde{\omega}_n + \omega) - \text{sgn}(\omega) = 2\text{sgn}(\tilde{\omega}_n)\Theta(-\omega(\tilde{\omega}_n + \omega)) . \quad (16)$$

Then it is easy to show that

$$\Sigma_1^\alpha(k_\parallel, i\tilde{\omega}_n)_{(q > k_0)} = \frac{-i\text{sgn}(\tilde{\omega}_n)}{\pi m^*} \int_{k_0}^\kappa dq q \ln\left(1 + \frac{|\tilde{\omega}_n|}{\Gamma_q}\right) . \quad (17)$$

Introducing the new integration variable $y = \Gamma_q/|\tilde{\omega}_n|$ and dimensionless frequencies $\bar{\omega}_n = \tilde{\omega}_n/(v_F q_c)$ and wave-vectors $\bar{k}_\parallel = k_\parallel/q_c$, we have

$$q = q_c |\bar{\omega}_n y|^{\frac{1}{1+\eta}} , \quad dq = q_c |\bar{\omega}_n|^{\frac{1}{1+\eta}} \frac{y^{\frac{1-\eta}{1+\eta}}}{1+\eta} dy , \quad (18)$$

so that

$$\frac{\Sigma_1^\alpha(k_\parallel, i\tilde{\omega}_n)_{(q > k_0)}}{v_F q_c} = -ig \text{sgn}(\tilde{\omega}_n) |\bar{\omega}_n|^{\frac{2}{1+\eta}} F_1(\bar{k}_\parallel, i\bar{\omega}_n) , \quad (19)$$

where $g = q_c/k_F \ll 1$ is a dimensionless coupling constant. The dimensionless function $F_1(\bar{k}_\parallel, i\bar{\omega}_n)$ is defined by

$$F_1(\bar{k}_\parallel, i\bar{\omega}_n) = \frac{1}{\pi(1+\eta)} \int_{y_0}^{y_1} dy y^{\frac{1-\eta}{1+\eta}} \ln(1 + 1/y) , \quad (20)$$

with the lower limit

$$y_0 = \frac{\Gamma_{k_0}}{|\tilde{\omega}_n|} = \left(\frac{k_0}{q_c}\right)^\eta \frac{v_F k_0}{|\tilde{\omega}_n|} = \begin{cases} |\bar{\omega}_n|^\eta & \text{if } |\bar{\omega}_n| \geq |\bar{k}_\parallel| \\ |\bar{k}_\parallel|^{1+\eta} |\bar{\omega}_n|^{-1} & \text{if } |\bar{\omega}_n| \ll |\bar{k}_\parallel| \end{cases} , \quad (21)$$

and the upper limit

$$y_1 = \frac{\Gamma_\kappa}{|\tilde{\omega}_n|} = \frac{1}{|\bar{\omega}_n|} \left(\frac{\kappa}{q_c} \right)^{1+\eta}. \quad (22)$$

Note that for large y the integrand in Eq.(20) vanishes as $y^{\frac{1-\eta}{1+\eta}-1}$, so that for $\eta > 1$ the integral is ultraviolet convergent. To obtain the limiting behavior of the self-energy for $|\bar{\omega}_n| \rightarrow 0$, we may let $y_1 \rightarrow \infty$. Furthermore, for $|\bar{\omega}_n| \geq |\bar{k}_\parallel|$ we may set $y_0 = 0$ in the lower limit of Eq.(20), so that

$$F_1(\bar{k}_\parallel, i\bar{\omega}_n) \approx \frac{1}{\pi(1+\eta)} \int_0^\infty dy y^{\frac{1-\eta}{1+\eta}} \ln(1+1/y) \equiv c_\eta. \quad (23)$$

Hence, to leading order

$$\frac{\Sigma_1^\alpha(k_\parallel, i\tilde{\omega}_n)_{(q>k_0)}}{v_F q_c} \sim -ic_\eta g \operatorname{sgn}(\tilde{\omega}_n) |\bar{\omega}_n|^{\frac{2}{1+\eta}}, \quad |\bar{\omega}_n| \geq |\bar{k}_\parallel|. \quad (24)$$

For completeness, let us also discuss the regime $|\bar{\omega}_n| \ll |\bar{k}_\parallel|$, where $y_0 \gg 1$. Then

$$F_1(\bar{k}_\parallel, i\bar{\omega}_n) = \frac{1+\eta}{\eta-1} y_0^{-1} + O(y_0^{-2}), \quad (25)$$

so that

$$\frac{\Sigma_1^\alpha(k_\parallel, i\tilde{\omega}_n)_{(q>k_0)}}{v_F q_c} \sim -\frac{g}{\pi(\eta-1)} \frac{i\bar{\omega}_n |\bar{\omega}_n|^{\frac{2}{1+\eta}}}{|\bar{k}_\parallel|^{1+\eta}}, \quad |\bar{\omega}_n| \ll |\bar{k}_\parallel|. \quad (26)$$

The spectral function can now be obtained from the retarded self-energy $\Sigma_1^\alpha(k_\parallel, \omega + i0^+)$,

$$A(k_\parallel, \omega) = -\frac{1}{\pi} \operatorname{Im} \left[\frac{1}{\omega - \xi_{k_\parallel} - \Sigma_1^\alpha(k_\parallel, \omega + i0^+)} \right]. \quad (27)$$

To perform the analytic continuation (i.e. $i\tilde{\omega}_n \rightarrow \omega + i0^+$, $\operatorname{sgn}(\tilde{\omega}_n) \rightarrow 1$, and $|\tilde{\omega}_n| \rightarrow -i\omega$) we write

$$i\operatorname{sgn}(\tilde{\omega}_n) |\bar{\omega}_n|^{\frac{2}{1+\eta}} = i\tilde{\omega}_n [-i\operatorname{sgn}(\tilde{\omega}_n) i\tilde{\omega}_n]^{\frac{1-\eta}{1+\eta}}, \quad (28)$$

so that

$$\begin{aligned} i\operatorname{sgn}(\tilde{\omega}_n) |\bar{\omega}_n|^{\frac{2}{1+\eta}} &\rightarrow \operatorname{sgn}(\omega) |\omega|^{\frac{2}{1+\eta}} \exp \left[i\operatorname{sgn}(\omega) \frac{\pi}{2} \frac{\eta-1}{\eta+1} \right] \\ &= |\omega|^{\frac{2}{1+\eta}} [\lambda' \operatorname{sgn}(\omega) + i\lambda''], \end{aligned} \quad (29)$$

with $\lambda' = \cos(\frac{\pi}{2} \frac{\eta-1}{\eta+1})$ and $\lambda'' = \sin(\frac{\pi}{2} \frac{\eta-1}{\eta+1})$. Note that for $\eta > 1$ both λ' and λ'' are positive real constants. We conclude that the retarded self-energy is for small frequencies in the regime $|\bar{\omega}| \geq |\bar{k}_\parallel|$ given by

$$\frac{\Sigma_1^\alpha(k_\parallel, \omega + i0^+)}{v_F q_c} \sim -c_\eta g |\bar{\omega}|^{\frac{2}{1+\eta}} [\lambda' \operatorname{sgn}(\omega) + i\lambda'']. \quad (30)$$

Note that $\text{Im}\Sigma_1^\alpha(k_\parallel, \omega + i0^+) < 0$, so that the retarded Green's function is analytic in the entire upper half of the complex frequency plane (as it should), and the corresponding spectral function (27) is positive (as it should). However, the spectral function does not exhibit a well-defined quasi-particle peak, because according to Eq.(30) the imaginary part of the self-energy has the same order of magnitude as the real part. Thus, lowest order perturbation theory suggests that fluctuations of the gauge field completely destroy the Fermi liquid state predicted by mean field theory.

From our rather detailed derivation it is now easy to identify the regions in energy-momentum space that are responsible for this non-Fermi liquid behavior. Obviously, in deriving Eq.(30) we may restrict ourselves to the regime where $|W(q, \omega)|$ is small compared with unity, which requires $k_0 \lesssim q$. Moreover, the leading term in the expansion of the function $Z(W)$ (see Eq.(14)) for small $|W|$ is due to angles ϑ in the integral representation (10) of this function such that $|\cos \vartheta| \lesssim |W| \ll 1$. This is most easily seen by shifting $x = \vartheta - \pi/2$ in Eq.(10), and using the fact that for small $|W|$ the leading behavior of the integral is determined by $|x - \text{Re}W| \lesssim |\text{Im}W| \ll 1$. Hence,

$$Z(W) \approx -\frac{1}{\pi} \int_{-\pi/2}^{\pi/2} dx \frac{1}{x - \text{Re}W - i\text{Im}W}. \quad (31)$$

Using $\text{Im}(x \pm ie)^{-1} = \mp i\pi \text{sgn}(\epsilon)$, we see that $Z(W) \approx -i\text{sgn}\text{Im}(W)$ for small $|W|$, in agreement with Eq.(14).

The result (30) remains correct if we sum in addition all diagrams where the gauge field propagators do not intersect (thus neglecting vertex corrections). This is formally achieved by replacing the non-interacting Green's function G_0^α on the right-hand side of Eq.(2) by the one-loop renormalized Green's function G_1^α , thus turning Eq.(2) into an integral equation. Assuming that the momentum-dependence of the so-defined self-consistent self-energy $\tilde{\Sigma}_1^\alpha$ is again negligible, the expression of $\tilde{\Sigma}_1^\alpha$ is still of the form (9), except that $W(q, \omega)$ should be replaced by

$$\tilde{W}(q, \omega) = \frac{G_0^{-1} + i\omega - \tilde{\Sigma}_1^\alpha(i\tilde{\omega}_n + i\omega) - \frac{q^2}{2m^\alpha}}{v_F q(1 + \frac{k_\parallel}{k_c})}. \quad (32)$$

Because $\text{sgn}\text{Im}(W) = \text{sgn}\text{Im}(\tilde{W})$, the resulting self-consistent self-energy is again given by Eq.(15), but with the lower limit k_0 replaced by $\tilde{k}_0 \approx q_c |\bar{\omega}_n|^{\frac{2}{1+n}}$, so that $|\tilde{W}| \lesssim 1$. This leads to the new lower limit $\tilde{y}_0 = |\bar{\omega}_n|$ in Eq.(20). Obviously, for $|\bar{\omega}_n| \rightarrow 0$ we may still set $\tilde{y}_0 = 0$. Thus, higher order diagrams without crossings of the gauge field lines do not modify the result predicted by lowest order perturbation theory.

Let us summarize what we have learned so far. Within a perturbative calculation to first order in the RPA screened gauge field propagator, one finds that the fluctuations of the transverse gauge field give rise to a non-analytic contribution to the fermionic self-energy, implying the non-existence of well defined quasi-particles. Denoting by \mathbf{q} the momentum transfer and by ω the energy transfer mediated by

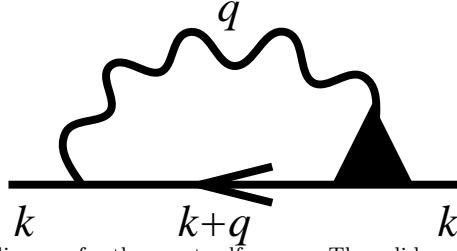


Fig. 3. Skeleton diagram for the exact self-energy. The solid arrow is the exact Green's function, the wavy line is the exact screened gauge field propagator, and the solid triangle represents the exact three-legged vertex.

the gauge field, the non-analyticity in the self-energy arises from energy-momentum transfers in the regime

$$|\cos \vartheta| \lesssim \frac{|\omega|}{v_F q} \ll 1, \quad (33)$$

where ϑ is the angle between \mathbf{q} and the local Fermi velocity \mathbf{v}^α . In other words, the non-Fermi liquid behavior is entirely due to momentum transfers that are almost parallel to the Fermi surface, corresponding to $|\cos \vartheta| \approx |\vartheta \pm \frac{\pi}{2}| \ll 1$. As will be discussed in Sec.3.2, this greatly reduces the usefulness of the well-known Ward-identities for estimating the importance of vertex corrections.

3. Vertex corrections

3.1. General remarks

Given the fact that the leading fluctuation correction qualitatively modifies the mean field result, one should worry about higher order corrections. This problem has been addressed previously by Altshuler, Ioffe, and Millis⁸ (AIM), and by Stern and Halperin²¹. We shall comment on these works below.

The general structure of the self-energy is conveniently represented in terms of the skeleton diagram shown in Fig.3. The corresponding analytic expression is

$$\Sigma^\alpha(\tilde{k}) = -\frac{1}{\beta V} \sum_{\tilde{q}} h_{\tilde{q}}^\alpha \Lambda^\alpha(\tilde{k}; \tilde{q}) G^\alpha(\tilde{k} + \tilde{q}), \quad (34)$$

where $h_{\tilde{q}}^\alpha$ is the exact propagator of the gauge field, $G^\alpha(\tilde{k})$ is the exact Green's function, and $\Lambda^\alpha(\tilde{k}, \tilde{q})$ is the exact three-legged vertex. Obviously, three physically different types of corrections can be distinguished. First of all, there are corrections to the gauge field propagator beyond the RPA. Because the interaction mediated by the gauge field is most singular for small momentum transfers, the closed loop theorem discussed in Refs.^{19,22,23} guarantees that these corrections are small and can be safely ignored. This has been explicitly verified at two-loop order by Kim *et al.*²⁴. The second type of corrections consists of diagrams without crossings of gauge field lines; these contribute to the exact Green's function $G^\alpha(\tilde{k} + \tilde{q})$ on the right-hand

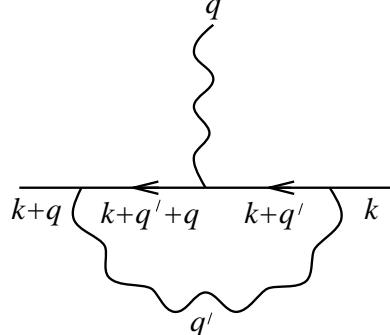


Fig. 4. Leading radiative correction to the fermion - gauge field vertex. The notation is the same as in Fig.1.

side of Eq.(34). At the end of Sec.2 we have shown that diagrams of this type do not modify the leading infrared behavior of the one-loop self-energy. However, it is a priori not clear whether this remains true for higher order diagrams involving more than one loop. In fact, according to Refs.^{7,8} it is essential to take these corrections into account by replacing $G_0 \rightarrow G_1$ in internal loops of higher order diagrams. We shall come back to this point below. The third type of diagrams are the vertex corrections, which are by definition all diagrams contributing to the vertex function $\Lambda^\alpha(\tilde{k}; \tilde{q})$. Naively, one could try to expand the vertex function in powers of the RPA gauge field propagator,

$$\Lambda^\alpha(\tilde{k}; \tilde{q}) = 1 + \sum_{n=1}^{\infty} \Lambda_n^\alpha(\tilde{k}; \tilde{q}), \quad (35)$$

where Λ_n^α is the sum of all contributions to the three-legged vertex involving n powers of $h^{\text{RPA},\alpha}$. However, this expansion might be ill-defined, because successive powers might be more and more singular. In this case non-perturbative methods are necessary to resum the perturbation series. Here we shall restrict ourselves to the more modest task of evaluating the leading vertex correction Λ_1^α shown in Fig.4, which is explicitly given by

$$\Lambda_1^\alpha(\tilde{k}; \tilde{q}) = -\frac{1}{\beta V} \sum_{\tilde{q}'} h_{\tilde{q}'}^{\text{RPA},\alpha} G_0^\alpha(\tilde{k} + \tilde{q}') G_0^\alpha(\tilde{k} + \tilde{q} + \tilde{q}'). \quad (36)$$

An attempt to evaluate Eq.(36) has been made in Ref.⁸. However, as will be explained in Sec.3.3, these authors have missed the dominant contribution. Before embarking on a careful evaluation of Eq.(36), let us explain with the help of Ward-identities why vertex corrections can be expected to play an important role in the present problem.

3.2. Ward-identities

In the limit of vanishing energy-momentum transfer $\tilde{q} = [\mathbf{q}, i\omega]$, the three-legged vertex $\Lambda^\alpha(\tilde{k}; \mathbf{q}, i\omega)$ can be related to appropriate partial derivatives of the self-energy

$\Sigma^\alpha(\tilde{k})$. In the *dynamic limit* we first set $\mathbf{q} = 0$ and then let $\omega \rightarrow 0$ (see Ref.²⁵). The corresponding Ward-identity reads in our notation

$$\Lambda_{\text{dy}}^\alpha(\tilde{k}) \equiv \lim_{\omega \rightarrow 0} \Lambda^\alpha(\tilde{k}; 0, i\omega) = 1 - \frac{\partial \Sigma^\alpha(\tilde{k})}{\partial(i\tilde{\omega}_n)}. \quad (37)$$

On the other hand, in the *static limit*, we set first $\omega = 0$ and then take the limit $\mathbf{q} \rightarrow 0$. This yields

$$\Lambda_{\text{stat}}^\alpha(\tilde{k}) \equiv \lim_{\mathbf{q} \rightarrow 0} \Lambda^\alpha(\tilde{k}; \mathbf{q}, 0) = 1 + \frac{\partial \Sigma^\alpha(\tilde{k})}{\partial(v_F k_\parallel)}. \quad (38)$$

It has been argued²¹ that these Ward-identities imply that vertex corrections are negligible in the present problem. The argument is based on the observation that the gauge field propagator (4) is singular in the static limit $|\omega| \ll v_F q$, so that at the first sight it seems that the vertex correction should be estimated from the static Ward-identity (38). Because the first order self-energy (24) is independent of k_\parallel , this would imply that the vertex $\Lambda^\alpha(\tilde{k}; \tilde{q})$ in Eq.(34) can be safely replaced by unity. However, this argument is incorrect, because it ignores the fact that the non-analyticity of the first order self-energy is entirely due to momentum transfers that are essentially parallel to the Fermi surface. To see this more clearly, let us recall that for interactions with dominant forward scattering the following more general Ward-identity can be derived²³,

$$[i\omega - \mathbf{v}^\alpha \cdot \mathbf{q}] \Lambda^\alpha(\tilde{k}; \tilde{q}) = \frac{1}{G^\alpha(\tilde{k} + \tilde{q}/2)} - \frac{1}{G^\alpha(\tilde{k} - \tilde{q}/2)}. \quad (39)$$

The dynamic Ward-identity Eq.(37) can be obtained as a special case of Eq.(39) by taking the limits $\omega \rightarrow 0$ and $\mathbf{q} \rightarrow 0$ such that the ratio

$$r \equiv \frac{\mathbf{v}^\alpha \cdot \mathbf{q}}{|\omega|} = \frac{v_F q \cos \vartheta}{|\omega|} \quad (40)$$

vanishes²³. Similarly, the static Ward-identity (38) is obtained by taking these limits such that $r \rightarrow \infty$. The crucial point is now that according to Eq.(33) the ratio $v_F q \cos \vartheta / |\omega|$ is *small compared with unity* for the relevant energy-momentum transfers that are responsible for the non-analyticities in the first-order self-energy. Hence, the vertex function should be estimated from the *dynamic* Ward-identity (37). If we now substitute the perturbative self-energy (24) into the right-hand side of Eq.(37), we see that the vertex actually diverges as $|\tilde{\omega}_n|^{\frac{1-\eta}{1+\eta}}$ for $\tilde{\omega}_n \rightarrow 0$ (recall that $\eta > 1$). It should be kept in mind, however, that the Ward-identity (39) has been derived by linearizing the energy dispersion in the vicinity of the Fermi surface, and therefore does contain information about *curvature* effects. As already pointed out in Refs.^{7,8} and discussed in detail below, the curvature of the Fermi surface plays a crucial role in the present problem, so that the Ward-identities (37–39) *do not have much predictive power*. In particular, for a spherical Fermi surface the

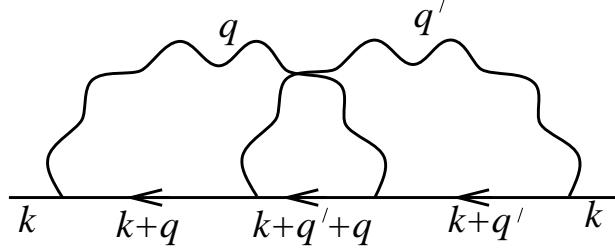


Fig. 5. Leading vertex correction to the self-energy.

above Ward-identities *cannot be used to obtain qualitative estimates for the order of magnitude of vertex corrections.* We shall verify the correctness of this statement in the next section by explicitly evaluating the leading vertex correction.

3.3. The leading vertex correction to the self-energy

The vertex correction Λ_1^α in Eq.(36) gives rise to the following two-loop correction to the self-energy,

$$\Sigma_2^\alpha(\tilde{k}) = -\frac{1}{\beta V} \sum_{\tilde{q}} h_{\tilde{q}}^\alpha \Lambda_1^\alpha(\tilde{k}; \tilde{q}) G_0^\alpha(\tilde{k} + \tilde{q}). \quad (41)$$

The corresponding Feynman diagram is shown in Fig.5. Note that in two dimensions the vertex $\Lambda_1^\alpha(\tilde{k}; \tilde{q})$ is a rather complicated function of six variables. From the evaluation of the one-loop self-energy we expect that the regime where the momentum transfers \mathbf{q} and \mathbf{q}' are almost parallel to the Fermi surface will play an important role. Below we shall first reproduce the result of AIM⁸, and then show that these authors have missed the dominant contribution. The reason is rather subtle: it turns out that the dominant contribution to the *self-energy* Σ_2^α is determined by a *sub-dominant* contribution to the *vertex* Λ_1^α .

3.3.1. Exact manipulations

We begin with the calculation of the function $\Lambda_1^\alpha(\tilde{k}; \tilde{q})$. Setting for simplicity $\mathbf{k} = k_\parallel \hat{\mathbf{v}}^\alpha$ and introducing the circular coordinates shown in Fig.6, we obtain

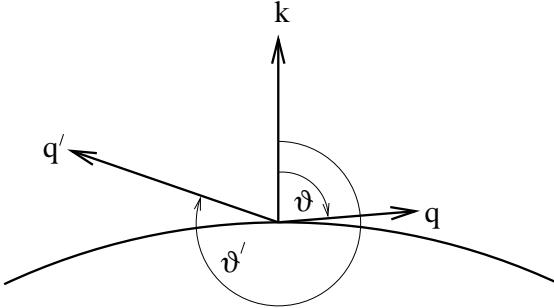
$$\begin{aligned} \Lambda_1^\alpha(k_\parallel, i\tilde{\omega}_n; q, \vartheta, i\omega) &= \\ \frac{1}{2\pi v_F m^*(1 + \frac{k_\parallel}{k_c})(1 + \frac{k_\parallel + q \cos \vartheta}{k_c})} &\int_0^\kappa dq' \int_{-v_F q'}^{v_F q'} d\omega' \frac{I(W', U, \Delta)}{\Gamma_{q'} + |\omega'|}, \end{aligned} \quad (42)$$

with the dimensionless integral $I(W', U, \Delta)$ given by

$$I(W', U, \Delta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\vartheta' \frac{\sin^2 \vartheta'}{(W' - \cos \vartheta')(U - \cos \vartheta' - \Delta \sin \vartheta')}, \quad (43)$$

and

$$W' = \frac{G_0^{-1} + i\omega' - \frac{q'^2}{2m^\alpha}}{v_F q'(1 + \frac{k_\parallel}{k_c})}, \quad (44)$$

Fig. 6. Definition of the angles ϑ and ϑ' .

$$U = \frac{G_0^{-1} + i\omega' + i\omega - (1 + \frac{k_{\parallel}}{k_c})v_F q \cos \vartheta - \frac{q^2 + q'^2}{2m^{\alpha}}}{v_F q'(1 + \frac{k_{\parallel} + q \cos \vartheta}{k_c})}, \quad (45)$$

$$\Delta = \frac{q}{k_c} \frac{\sin \vartheta}{(1 + \frac{k_{\parallel} + q \cos \vartheta}{k_c})}. \quad (46)$$

The angular integration in Eq.(43) can be performed exactly. The result is

$$\begin{aligned} I(W', U, \Delta) &= \frac{4}{(1 + W')(1 + U)[(B - A)^2 + AC^2]} \\ &\times \left\{ (B - A) \left[\frac{1 - \sqrt{A}}{1 - A} - \frac{(1 - B)(1 - \sqrt{B - C^2/4})}{(1 - B)^2 + C^2} \right] \right. \\ &\left. + \frac{C^2}{(1 - B)^2 + C^2} \left[-B + \frac{(B + A)(1 + B)}{4\sqrt{B - C^2/4}} \right] \right\}, \end{aligned} \quad (47)$$

where we have defined

$$A = \frac{W' - 1}{W' + 1}, \quad B = \frac{U - 1}{U + 1}, \quad C = \frac{2\Delta}{U + 1}, \quad (48)$$

and the roots have to be taken such that $\text{Re}\sqrt{A} \geq 0$ and $\text{Re}\sqrt{B - C^2/4} \geq 0$. To make further progress without resorting to numerical methods, the obvious strategy is to identify the regimes in the $q' - \omega'$ -plane which dominate the integral and then find some simplification of the integrand such that the integration can be carried out analytically. This is actually not so easy, because we are eventually interested in the self-energy (41), and a priori we cannot exclude the possibility that, due to certain symmetry-related cancellations, sub-dominant contributions to the vertex Λ_1^{α} are responsible for the dominant infrared behavior of the self-energy Σ_2^{α} . In fact, we shall show shortly that this is precisely what happens.

3.3.2. The contribution from the regime $q/q' = O(1)$

Let us for the moment assume that the leading infrared behavior of the self-energy Σ_2^{α} in Eq.(41) is determined by the regime where both parameters $|W'|$ and $|U|$ are

small compared with unity. At the first sight this assumption seems reasonable, because in this case the integrand in Eq.(43) has two poles in the vicinity of the real axis, so that the value of the integral is large compared with unity. We now show that with this assumption we can reproduce the result of AIM⁸. Later we shall demonstrate that this assumption is *not correct*: the infrared behavior of the self-energy in Eq.(41) is in fact determined by the regime where $|U| \gg 1$!

Obviously, the condition $|W'| \lesssim 1$ is equivalent with (see also Eq.(12))

$$q' \gtrsim k_0 = \max \{ |k_{\parallel}|, |\tilde{\omega}_n|/v_F \} , \quad (49)$$

and $|U| \lesssim 1$ requires

$$q' \gtrsim q_0 = \max \{ k_0, |\omega|/v_F, q^2/k_c \} , \quad (50)$$

where we have used the fact that according to Eq.(33) $|\omega| \gtrsim v_F q |\cos \vartheta|$. In this regime it is easy to show from Eq.(47) that to leading order

$$\begin{aligned} I(W', U, \Delta) &\approx \frac{i}{2} [\operatorname{sgn}(\operatorname{Im}U) - \operatorname{sgn}(\operatorname{Im}W')] \\ &\times \left[\frac{1}{U - W' - \Delta} + \frac{1}{U - W' + \Delta} \right] . \end{aligned} \quad (51)$$

Actually, Eq.(51) can be derived in a much simpler way from Eq.(43) if we use the fact that for small $|W'|$ and $|U|$ the integral is dominated by $\vartheta' \approx \pm\pi/2$. Then we may substitute $\vartheta' = x' \pm \pi/2$, replace $\sin(x' \pm \pi/2) \approx \pm 1$, and proceed as in Eq.(31). From Eqs.(44) and (45) we find to leading order

$$U - W' = \frac{i\omega - v_F q \cos \vartheta - \frac{q^2}{2m^\alpha}}{v_F q'} , \quad (52)$$

and

$$\operatorname{sgn}(\operatorname{Im}U) - \operatorname{sgn}(\operatorname{Im}W') = \operatorname{sgn}(\tilde{\omega}_n + \omega + \omega') - \operatorname{sgn}(\tilde{\omega}_n + \omega') . \quad (53)$$

Substituting Eqs.(51–53) into Eq.(42), we obtain after a straightforward calculation

$$\begin{aligned} \Lambda_1^\alpha(k_{\parallel}, i\tilde{\omega}_n; q, \vartheta, i\omega)_{(q' > q_0)} &= \\ \frac{-i}{2\pi m^*} \int_{q_0}^{\infty} dq' q' &\left[\operatorname{sgn}(\tilde{\omega}_n) \ln \left(1 + \frac{|\tilde{\omega}_n|}{\Gamma_{q'}} \right) - \operatorname{sgn}(\tilde{\omega}_n + \omega) \ln \left(1 + \frac{|\tilde{\omega}_n + \omega|}{\Gamma_{q'}} \right) \right] \\ &\times \left[\frac{1}{i\omega - v_F q \cos \vartheta - \frac{q^2}{2m^\alpha} - \frac{qq'}{m^\alpha} \sin \vartheta} + \frac{1}{i\omega - v_F q \cos \vartheta - \frac{q^2}{2m^\alpha} + \frac{qq'}{m^\alpha} \sin \vartheta} \right] . \end{aligned} \quad (54)$$

The subscript $(q' > q_0)$ indicates that this is the contribution from the regimes (49) and (50). Note that by definition $q_0 > k_0$.

Let us pause for a moment and compare Eq.(54) with the considerations of Sec.3.2. Because the Ward-identity (39) ignores the non-linear terms in the energy dispersion, let us take the limit $1/m^\alpha \rightarrow 0$ in Eq.(54). Comparing the q' -integral

with the corresponding integral in the expression for Σ_1^α (see Eq.(17)), and taking into account that according to Eq.(33) $|\omega| \gtrsim v_F q |\cos \vartheta|$ and that the integral is dominated by frequencies $|\omega| \lesssim |\tilde{\omega}_n|$, it is easy to see that for linearized energy dispersion Λ_1^α is indeed proportional to $|\tilde{\omega}_n|^{\frac{1-\eta}{1+\eta}}$. This is in agreement with the arguments presented in Sec.3.2: in the limit $1/m^\alpha \rightarrow 0$ the order of magnitude of the vertex correction can be obtained from the *dynamic* Ward-identity (37), and not from the static Ward-identity (38).

However, the curvature terms in the denominator of Eq.(54) cannot be neglected! Substituting Eq.(54) into Eq.(41), we obtain

$$\begin{aligned} \Sigma_2^\alpha(k_{\parallel}, i\tilde{\omega}_n)_{(q' > q_0)} &= \frac{-i}{(2\pi)^3(m^*)^2 v_F} \int_0^\kappa dq \int_{-v_F q}^{v_F q} d\omega \frac{1}{\Gamma_q + |\omega|} \\ &\times \int_{q_0}^\kappa dq' q' \left[\operatorname{sgn}(\tilde{\omega}_n) \ln \left(1 + \frac{|\tilde{\omega}_n|}{\Gamma_{q'}} \right) - \operatorname{sgn}(\tilde{\omega}_n + \omega) \ln \left(1 + \frac{|\tilde{\omega}_n + \omega|}{\Gamma_{q'}} \right) \right] \\ &\times \int_{-\pi}^{\pi} d\vartheta \frac{\sin^2 \vartheta}{W - \cos \vartheta} \left[\frac{1}{W_0 - \cos \vartheta - \frac{q'}{k_c} \sin \vartheta} + \frac{1}{W_0 - \cos \vartheta + \frac{q'}{k_c} \sin \vartheta} \right], \end{aligned} \quad (55)$$

where W is defined in Eq.(8), and $W_0 = i\omega/(v_F q) - q/(2k_c)$. The angular integration is the same as in Eq.(43), and can be done exactly. For $|W| \ll 1$ (which requires $q > k_0$, see Eq.(12)) the integral is dominated by $\vartheta \approx \pm\pi/2$, so that we may again set $\sin \vartheta \approx \pm 1$ and extract the leading behavior of the integral as in Eq.(31). We obtain

$$\begin{aligned} \Sigma_2^\alpha(k_{\parallel}, i\tilde{\omega}_n)_{(q' > q_0)} &= \frac{-G_0^{-1}}{2\pi^2(m^*)^2} \int_{k_0}^\kappa dq q \int_0^{|\tilde{\omega}_n|} d\omega \frac{1}{\Gamma_q + \omega} \\ &\times \int_{q_0}^\kappa dq' q' \ln \left[1 - \frac{\omega}{\Gamma_{q'} + |\tilde{\omega}_n|} \right] \frac{1}{(G_0^{-1})^2 - (\frac{qq'}{m^\alpha})^2}. \end{aligned} \quad (56)$$

In the limit $1/m^\alpha \rightarrow 0$ the term qq'/m^α in the denominator of Eq.(56) vanishes, so that the integral is proportional to $G_0 = [i\tilde{\omega}_n - \xi_{k_{\parallel}}]^{-1}$. Then the dependence on $\tilde{\omega}_n$ can then be scaled out, and we find $\Sigma_2^\alpha/\Sigma_1^\alpha \propto |\tilde{\omega}_n|^{\frac{2}{1+\eta}} / (i\tilde{\omega}_n - \xi_{k_{\parallel}})$. This rather singular result is drastically modified by the curvature term qq'/m^α . This is most easily seen by noting that according to Eq.(18) the term qq'/m^α scales as $|\tilde{\omega}_n|^{\frac{2}{1+\eta}}$, which for $\eta > 1$ is much larger than G_0^{-1} . To leading order we obtain from Eq.(56)

$$\Sigma_2^\alpha(k_{\parallel}, i\tilde{\omega}_n)_{(q' > q_0)} \propto \left(\frac{m^\alpha}{m^*} \right)^2 [i\tilde{\omega}_n - \xi_{k_{\parallel}}] \ln^2 \left(\frac{q_c |\tilde{\omega}_n|^{\frac{2}{1+\eta}}}{k_0} \right). \quad (57)$$

For a circular Fermi surface, where $m^* = m^\alpha$, this agrees with Eq.(8) of AIM⁸. Obviously Eq.(57) is negligible compared with the first order self-energy given in Eq.(24).

Because according to Eq.(57) $|\Sigma_2| \ll |\Sigma_1|$, AIM argue that the first order self-energy correction should actually be included in all internal propagators. In other words, the free Green's functions G_0^α in Eqs.(36) and (41) should be replaced by

the one-loop corrected Green's function G_1^α . If one repeats the above calculation with this renormalized propagator, one finds⁸ $\Sigma_2^\alpha \propto |\tilde{\omega}_n|^{\frac{2}{1+\eta}}$. This is much larger than Eq.(57), and has the same order of magnitude as Σ_1^α . However, we shall show shortly that the dominant infrared behavior of the two-loop diagram shown in Fig.5 is not correctly given by Eq.(57), but is in fact logarithmically larger than the one-loop self-energy Σ_1^α . Consequently, Eq.(57) cannot be used to justify the infinite resummation of perturbation theory adopted in Ref.⁸, where all internal propagators in higher order diagrams are replaced by one-loop corrected propagators G_1^α . AIM further justify their approach by noting that the *reducible* diagram corresponding to the term $\Sigma_1 G_0 \Sigma_1$ in the first iteration of the Dyson equation (see Fig.2b of Ref.⁸) is larger than the *irreducible* diagram shown in Fig.5. While such a point of view might be valid within the framework of a large- N expansion, we do not believe that for small N the comparison of irreducible and reducible diagrams is physically meaningful: because reducible diagrams contain unphysical poles for frequencies close to the non-interacting energy dispersion, their contribution to the self-energy can always be tuned to be arbitrarily large.

3.3.3. The contribution from the regime $q \gg q'$

We now show that in the limit of vanishing frequency the leading infrared behavior of Σ_2^α defined in Eq.(41) is not given by Eq.(57), but that the two-loop self-energy (41) in fact exceeds the one-loop self-energy Σ_1^α by a logarithmically divergent factor,

$$\frac{\Sigma_2^\alpha(k_\parallel, i\tilde{\omega}_n)}{\Sigma_1^\alpha(k_\parallel, i\tilde{\omega}_n)} \sim -a_\eta \ln(1/|\bar{\omega}_n|) \quad \text{for } |\bar{\omega}_n| \rightarrow 0, \quad (58)$$

where a_η is a positive numerical constant proportional to m^α/m^* . The key observation is that so far we have tacitly assumed that the leading infrared behavior of Eq.(41) is completely determined by the regime where $|U| \lesssim 1$, which means that $q' \gtrsim q_0 = \max\{k_0, |\omega|/v_F, q^2/k_c\}$, see Eq.(50). Let us now check the contribution from the opposite regime $|U| \gtrsim 1$, equivalent with $q' \lesssim q_0$. From Eq.(47) (or more simply from Eq.(43)) we see that in this case

$$I(W', U, \Delta) \approx \frac{Z(W')}{U} \quad , \quad |U| \gtrsim 1, \quad (59)$$

with $Z(W')$ defined in Eq.(10). Note that $|U| \gtrsim 1$ implies $|\omega| \gtrsim v_F q'$, and hence $q \gg q'$ (keeping in mind that $v_F q \gtrsim |\omega|$). Because by construction $|W'| \ll 1$, it is clear from Eqs.(45) and (59) that in this regime

$$I(W', U, \Delta) \approx \frac{-i \operatorname{sgn}(\tilde{\omega}_n + \omega') v_F q'}{G_0^{-1} + i\omega - v_F q \cos \vartheta - \frac{q^2}{2m^\alpha}} \quad , \quad q' \lesssim q_0. \quad (60)$$

Substituting this expression into Eq.(42) and performing the ω' -integration, we see that the contribution from the regime $q' \lesssim q_0$ to the vertex Λ_1^α is

$$\Lambda_1^\alpha(k_\parallel, i\tilde{\omega}_n; q, \vartheta, i\omega)_{(q' < q_0)} = \frac{-i}{\pi m^*} \frac{\operatorname{sgn}(\tilde{\omega}_n)}{G_0^{-1} + i\omega - v_F q \cos \vartheta - \frac{q^2}{2m^\alpha}}$$

$$\times \int_{k_0}^{q_0} dq' q' \ln \left(1 + \frac{|\tilde{\omega}_n|}{\Gamma_{q'}} \right) . \quad (61)$$

Let us now assume that $|\omega| \gtrsim \Omega(q, \vartheta)$, where

$$\Omega(q, \vartheta) \equiv \max\{v_F q \cos \vartheta, \frac{q^2}{2m^\alpha}, v_F q_c |\bar{\omega}_n|^{\frac{1}{1+\eta}}\} . \quad (62)$$

The condition $|\omega| \geq v_F q_c |\bar{\omega}_n|^{\frac{1}{1+\eta}}$ allows us to neglect the term G_0^{-1} in the denominator of Eq.(61) and to replace the upper cutoff of the q' -integral by infinity. Introducing again the rescaled frequencies $\bar{\omega}_n = \tilde{\omega}_n / (v_F q_c)$ and $\bar{\omega} = \omega / (v_F q_c)$ (see Sec.2) we obtain in this regime from Eq.(61)

$$\Lambda_1^\alpha(k_\parallel, i\tilde{\omega}_n; q, \vartheta, i\omega)_{(q' < q_0)} \approx -ic_\eta g \operatorname{sgn}(\tilde{\omega}_n) \frac{|\bar{\omega}_n|^{\frac{2}{1+\eta}}}{i\bar{\omega}} , \quad |\omega| \gtrsim \Omega(q, \vartheta) , \quad (63)$$

where the numerical constant c_η is defined in Eq.(23). Because by assumption $g \equiv q_c/k_F \ll 1$ and $|\bar{\omega}| \gtrsim |\bar{\omega}_n|^{\frac{2}{1+\eta}}$, the numerical value of the vertex function (63) is certainly small compared with unity. Nevertheless, the contribution from this regime dominates the infrared behavior of the self-energy, because Eq.(63) is an *odd function of ω , and has therefore a different symmetry than the bare vertex*. Substituting Eq.(63) into Eq.(41) and using the fact that for $|\omega| \gtrsim \Omega(q, \vartheta)$ we may approximate $G_0^\alpha(\tilde{k} + \tilde{q}) \approx (i\omega)^{-1}$, we obtain

$$\begin{aligned} \Sigma_2^\alpha(k_\parallel, i\tilde{\omega}_n)_{(q' < q_0)} &\approx \frac{v_F}{(2\pi)^2 m^*} \int_0^\kappa dq q^2 \int_{-\pi}^\pi d\vartheta \sin^2 \vartheta \\ &\times \int_{-v_F q}^{v_F q} d\omega \frac{\Theta(|\omega| - \Omega(q, \vartheta))}{i\omega(\Gamma_q + |\omega|)} \Lambda_1^\alpha(k_\parallel, i\tilde{\omega}_n; q, \vartheta, i\omega)_{(q' < q_0)} . \end{aligned} \quad (64)$$

Next we note that for sufficiently small q the energy scale Γ_q is negligible compared with $q^2/(2m^\alpha)$, because Γ_q vanishes faster than q^2 . Let us define the wave-vector q_1 where both energy scales are identical,

$$\frac{q_1^2}{2m^\alpha} = \Gamma_{q_1} = v_F q_c \left(\frac{q_1}{q_c} \right)^{\frac{2}{1+\eta}} , \quad (65)$$

i.e. $q_1 = q_c (q_c/2k_c)^{\frac{1}{\eta-1}}$. It turns out that the leading contribution to Eq.(64) is due to the regime $q < q_1$. Because by construction $|\omega| > \Omega(q, \vartheta) \geq q^2/(2m^\alpha)$, we may then neglect the energy Γ_q compared with $|\omega|$ in the integrand. The ω -integration is now trivial and generates a factor of $\Omega^{-2}(q, \vartheta)$. Using the expression (24) for the one-loop self-energy Σ_1^α , we obtain

$$\Sigma_2^\alpha(k_\parallel, i\tilde{\omega}_n)_{(q' < q_0)} = \Sigma_1^\alpha(k_\parallel, i\tilde{\omega}_n) R(i\tilde{\omega}_n) , \quad (66)$$

with

$$R(i\tilde{\omega}_n) = -\frac{v_F}{(2\pi)^2 m^*} \int_0^{q_1} dq q^2 \int_{-\pi}^\pi d\vartheta \frac{\sin^2 \vartheta}{\Omega^2(q, \vartheta)} . \quad (67)$$

To leading logarithmic order the integration in Eq.(67) gives

$$R(i\tilde{\omega}_n) \sim -\frac{4}{\pi^2} \frac{m^\alpha}{m^*} \ln(q_1/q_2), \quad (68)$$

where q_2 is defined by

$$\frac{q_2^2}{2m^\alpha} = v_F q_c |\bar{\omega}_n|^{\frac{2}{1+\eta}}. \quad (69)$$

Using Eq.(65), we obtain for $|\bar{\omega}_n| \rightarrow 0$

$$R(i\tilde{\omega}_n) \sim -\frac{a_\eta}{2} \ln(1/|\bar{\omega}_n|), \quad (70)$$

where

$$a_\eta = \frac{8}{\pi^2(1+\eta)} \frac{m^\alpha}{m^*}. \quad (71)$$

Because of our rather crude method of estimating the integrals, the numerical value of a_η in Eq.(71) should not be taken too serious, and remains uncertain by a factor of the order of unity. Combining Eqs.(66), (70) and (71), and taking into account that by symmetry the regime $q' \gg q$ gives rise to an equally large contribution to the self-energy (this is obvious from the labels in Fig.5), we finally arrive at Eq.(58).

Due to the complexity of the integrations, we have not been able to check whether the result (58) is modified if the internal free propagators G_0^α in Fig.5 are replaced by one-loop corrected propagators G_1^α . However, as discussed at the end of Sec.3.3.2, we do not believe that such an infinite resummation of perturbation theory is a sensible procedure (except perhaps within a large- N expansion⁸).

4. Conclusions

In this work we have shown that at the two-loop order the low-frequency behavior of the self-energy of fermions that are coupled to transverse gauge fields with propagator of the form (4) is given by

$$\Sigma(i\tilde{\omega}_n) = -ic_\eta \frac{q_c^2}{m^*} \text{sgn}(\tilde{\omega}_n) \left| \frac{\tilde{\omega}_n}{v_F q_c} \right|^{\frac{2}{1+\eta}} \left[1 - a_\eta \ln \left(\frac{v_F q_c}{|\tilde{\omega}_n|} \right) \right], \quad (72)$$

where c_η and a_η are positive numerical constants. The logarithmic correction in Eq.(72) is due to the leading radiative correction to the fermion - gauge field vertex, and is the main result of this work. It implies that the correct infrared behavior of the self-energy of fermions that are coupled to transverse gauge fields cannot be obtained from a one-loop calculation. We disagree in this point with the authors of Refs.^{7,8}, who analyzed this problem within a suitably devised $1/N$ -expansion.

Because the leading vertex correction generates an additional logarithm, it seems likely that the higher order vertex corrections will give rise to even higher powers of logarithms. If we boldly exponentiate the logarithmic correction in Eq.(72), we find

$$\Sigma(i\tilde{\omega}_n) = -ic_\eta \frac{q_c^2}{m^*} \text{sgn}(\tilde{\omega}_n) \left| \frac{\tilde{\omega}_n}{v_F q_c} \right|^\gamma, \quad \gamma = \frac{2}{1+\eta} + a_\eta. \quad (73)$$

Note that according to Eq.(71) a_η is positive and of the order of unity for a spherical Fermi surface. It is therefore tempting to speculate that the summation of vertex corrections to all orders will indeed push the exponent γ to a value that is larger than predicted by the one-loop result. In fact, the non-perturbative calculation given in Ref.¹⁷ suggests that the true exponent γ is *not* smaller than unity, implying that the infrared fluctuations of the gauge field do *not* lead to a destruction of the Fermi liquid. This would explain why experimentally composite fermions in the half-filled Landau level seem to behave as well-defined quasi-particles²⁶.

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References

- [*] Address from 1 November 1997 – 15 April 1998.
- 1. G. Baskaran and P. W. Anderson, Phys. Rev. **B** 37, 580 (1988).
- 2. L. B. Ioffe and A. I. Larkin, Phys. Rev. **B** 39, 8988 (1989).
- 3. P. A. Lee, Phys. Rev. Lett. **63**, 680 (1989); N. Nagaosa and P. A. Lee, Phys. Rev. Lett. **64**, 2450 (1990).
- 4. B. I. Halperin, P. A. Lee, and N. Read, Phys. Rev. **B** 47, 7312 (1993).
- 5. J. Jain, Phys. Rev. Lett. **63**, 199 (1989); Phys. Rev. **B** 40, 8079 (1989); 41, 7653 (1990).
- 6. B. Blok and H. Monien, Phys. Rev. **B** 47, 3454 (1993).
- 7. L. B. Ioffe, D. Lidsky, and B. L. Altshuler, Phys. Rev. Lett. **73**, 472 (1994).
- 8. B. L. Altshuler, L. B. Ioffe, and A. J. Millis, Phys. Rev. **B** 50, 14048 (1994).
- 9. D. V. Khveshchenko and P. C. E. Stamp, Phys. Rev. Lett. **71**, 2118 (1993); Phys. Rev. **B** 49, 5227 (1994).
- 10. J. Gan and E. Wong, Phys. Rev. Lett. **71**, 4226 (1993).
- 11. J. Polchinski, Nucl. Phys. **B** 422, 617 (1994).
- 12. C. Nayak and F. Wilczek, Nucl. Phys. **B** 430, 534 (1994).
- 13. H. J. Kwon, A. Houghton, and J. B. Marston, Phys. Rev. Lett. **73**, 284 (1994); Phys. Rev. **B** 52, 8002 (1995).
- 14. M. Onoda, I. Ichinose, and T. Matsui, Nucl. Phys. **B** 446, 353 (1995); Phys. Rev. **B** 54, 13674 (1996).
- 15. S. Chakravarty, R. E. Norton, and O. Syljuasen, Phys. Rev. Lett. **75**, 3584 (1995).
- 16. P. Kopietz, Phys. Rev. **B** 53, 12761 (1996).
- 17. P. Kopietz and G. E. Castilla, Phys. Rev. Lett. **78**, 314 (1997).
- 18. P. Kopietz and G. E. Castilla, Phys. Rev. Lett. **76**, 4777 (1996).
- 19. P. Kopietz, *Bosonization of Interacting Fermions in Arbitrary Dimensions*, (Springer, Berlin, 1997).
- 20. The complex function $Z(W)$ defined in Eq.(10) is a particular inverse of the so-called *Joukowski map* $Z \rightarrow W = \frac{1}{2}(Z + \frac{1}{Z})$, which is uniquely defined by restricting the Z -domain to the interior of the unit circle, see P. Henrici, *Applied and Computational*

- Complex Analysis, Vol.1*, (Wiley Classics Library Edition, 1988), pp. 294–298.
- 21. A. Stern and B. I. Halperin, Phys. Rev. **B** 52, 5890 (1995).
 - 22. P. Kopietz, J. Hermisson, and K. Schönhammer, Phys. Rev. **B** 52, 10877 (1995).
 - 23. W. Metzner, C. Castellani, and C. Di Castro, preprint cond-mat/9702012, Adv. Phys. (in press).
 - 24. Y. B. Kim, A. Furusaki, X.-G. Wen, and P. A. Lee, Phys. Rev. **B** 50, 17917 (1994).
 - 25. P. Nozières, *Theory of Interacting Fermi Systems*, (Benjamin, New York, 1964).
 - 26. R. L. Willett *et al.*, Phys. Rev. Lett. **71**, 3846 (1993); W. Kang *et al.*, *ibid.* 3850 (1993); D. R. Leadley *et al.*, Phys. Rev. Lett. **72**, 1906 (1994); H. C. Manoharan *et al.*, Phys. Rev. Lett. **73**, 3270 (1994); R. R. Du *et al.*, Phys. Rev. Lett. **73**, 3274 (1994); P. T. Coleridge *et al.*, Phys. Rev. **B** 52, R11603 (1995); J. H. Smet *et al.*, Phys. Rev. Lett. **77**, 2272 (1996); R. L. Willett, Adv. Phys. **46**, 447 (1997).